

Recognizing $[h, 2, 1]$ graphs

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Abstract

An (h, s, t) -representation of a graph G consists of a collection of subtrees of a tree T , where each subtree corresponds to a vertex of G such that (i) the maximum degree of T is at most h , (ii) every subtree has maximum degree at most s , (iii) there is an edge between two vertices in the graph G if and only if the corresponding subtrees have at least t vertices in common in T . The class of graphs that have an (h, s, t) -representation is denoted $[h, s, t]$.

An undirected graph G is called a *VPT* graph if it is the vertex intersection graph of a family of paths in a tree. In this paper we characterize $[h, 2, 1]$ graphs using chromatic number. We show that the problem of deciding whether a given *VPT* graph belongs to $[h, 2, 1]$ is NP-complete, while the problem of deciding whether the graph belongs to $[h, 2, 1] - [h - 1, 2, 1]$ is NP-hard. Both problems remain hard even when restricted to $Split \cap VPT$. Additionally, we present a non-trivial subclass of $Split \cap VPT$ in which these problems are polynomial time solvable.

Key words: intersection graphs, VPT graphs, representations on trees, recognition problems.

1 Introduction

The intersection graph of a set family is a graph whose vertices are the members of the family, and the adjacency between them is defined by a

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non-empty intersection of the corresponding sets. Classical examples are interval graphs and chordal graphs.

An **interval graph** is the intersection graph of a family of closed intervals on the real line, or equivalently the intersection graph of a family of subpaths of a path. A **chordal graph** is a graph without induced cycles of length at least four. Gravril [4] proved that a graph is chordal if and only if it is the intersection graph of a family of subtrees of a tree, considering vertex intersection. Both classes have been widely studied [1].

In order to allow larger families of graphs to be represented by subtrees, several graph classes are defined imposing conditions on trees, subtrees and intersection sizes [11, 12]. An **(h, s, t) -representation** of a graph G consists of a collection of subtrees of a tree T , each subtree corresponding to a vertex of G , such that (i) the maximum degree of T is at most h , (ii) every subtree has maximum degree at most s , (iii) there is an edge between two vertices in the graph G if and only if the corresponding subtrees have at least t vertices in common in T . The class of graphs that have an (h, s, t) -representation is denoted $[h, s, t]$. When there is no restriction on the degree of T or on the degree of the subtrees, we use $h = \infty$ and $s = \infty$ respectively. Notice that $[\infty, \infty, 1]$ is the class of chordal graphs; $[2, 2, 1]$ is the class of interval graphs; $[\infty, 2, 1]$ and $[\infty, 2, 2]$ are the well known *VPT* and *EPT* graphs [14].

In [3], the minimum t such that a given graph belongs to $[3, 3, t]$ is studied. In [9], $[4, 4, 2]$ graphs are characterized and a polynomial time algorithm for their recognition is given. In [8], the class $[4, 2, 2]$ is studied. In [6], different aspects of $[\infty, 2, t]$ graphs are considered. The relation between the different classes is analyzed in [7]. In [5], it is shown that the problem of recognizing *VPT* graphs is polynomial time solvable, but the recognition of *EPT* graphs is an NP-complete problem.

In this work we focus on the classes $[h, 2, 1]$, all of them are subclasses of *VPT*. The problem is deciding whether a given *VPT* graph can be represented as intersection of paths in a tree with maximum degree h . Since $[2, 2, 1] = \text{Interval}$ and $[3, 2, 1] = [3, 2, 2] = \text{EPT} \cap \text{chordal}$ [5], we consider $h \geq 4$. We characterize $[h, 2, 1]$ graphs using chromatic number. We show that the problem of deciding whether a given *VPT* graph belongs to $[h, 2, 1]$ is NP-complete, while the problem of deciding whether the graph belongs to $[h, 2, 1] - [h - 1, 2, 1]$ is NP-hard. Both problems remain hard even when restricted to $\text{Split} \cap \text{VPT}$. Additionally, we present a non-trivial subclass of $\text{Split} \cap \text{VPT}$ in which these problems are polynomial time solvable. In Section 2, we provide basic definitions and known results. In Section 3, we characterize $[h, 2, 1]$ graphs for $h \geq 3$. In Section 4, we present the results about time complexity. Finally, in Section 5 we present some open questions.

2 Preliminaries

In this paper, all graphs are connected, finite, simple and loopless. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, the **open neighborhood** $\mathbf{N}_G(\mathbf{v})$ of a vertex v is the set of all vertices adjacent to v . The **closed neighborhood** $\mathbf{N}_G[\mathbf{v}]$ is $N_G(v) \cup \{v\}$. The **degree** of v , denoted by $\mathbf{d}_G(\mathbf{v})$, is the cardinality of $N_G(v)$. For simplicity, when no confusion arise, we omit the subindex G and simply write $N(v)$, $N[v]$ or $d(v)$.

For $S \subseteq V(G)$, $\mathbf{G}[\mathbf{S}]$ is the subgraph of G induced by S ; $\mathbf{G} - \mathbf{S}$ is a shorthand for $G[V(G) - S]$; and $\mathbf{G} - \mathbf{v}$ is used for $G - \{v\}$.

A **complete set** is a subset of vertices inducing a complete subgraph. A **clique** is a maximal complete set. The set of cliques of G is denoted by $\mathcal{C}(G)$. A **stable set** is a subset of vertices pairwise non-adjacent.

The graph G is **split** if $V(G)$ can be partitioned into a stable set S and a clique K [1]. The pair (\mathbf{S}, \mathbf{K}) is the **split partition** of G . The vertices in S are called **stable vertices**, and K is called the **central clique** of G . We say that a stable vertex $s \in S$ is **dominated** if there exists $s' \in S$ such that $N(s) \subseteq N(s')$. Notice that if G is split then $\mathcal{C}(G) = \{K, N[s] \text{ for } s \in S\}$.

A **VPT-representation** of G , denoted by $\langle \mathcal{P}, T \rangle$, is an $(\infty, 2, 1)$ -representation. This means that \mathcal{P} is a family $(P_v)_{v \in V(G)}$ of subpaths of a host tree T satisfying that two vertices v and v' of G are adjacent if and only if P_v and $P_{v'}$ have at least one vertex in common. If q is a vertex of the host tree T , then $\mathbf{P}[\mathbf{q}]$ denote the set $\{P \in \mathcal{P} / q \in V(P)\}$ and \mathbf{C}_q denote the complete set $\{v \in V(G) / q \in V(P_v)\}$. Notice that for every clique C of G , there exists $q \in V(T)$ such that $C = C_q$.

Definition 2.1 [5] *Let $C \in \mathcal{C}(G)$. The **branch graph** of G for the clique C denoted by $\mathbf{B}(\mathbf{G}/\mathbf{C})$ is defined as follows: the vertex set $V(\mathbf{B}(G/C))$ contains the vertices of $V(G) \setminus C$ adjacent to some vertex of C . Two vertices v and w are adjacent in $\mathbf{B}(G/C)$ if and only if*

1. $vw \notin E(G)$;
2. *there exists a vertex of C adjacent to both v and w ; and*
3. *there exist vertices v' and w' of C such that v' is adjacent to v and non-adjacent to w , and w' is adjacent to w and non-adjacent to v .*

Let $q \in V(T)$, with $N_T(q) = \{y_1, y_2, \dots, y_h\}$. We call **branches of \mathbf{T} at \mathbf{q}** to the connected components of $V(T) - \{q\}$. Observe that each y_i is contained in a different branch which will be called T_i .

The graph G is **k-colorable** if its vertices can be colored with at most k colors in such a way that no two adjacent vertices share the same color. The **chromatic number** of G , denoted by $\chi(G)$, is the smallest number of colors needed to coloring G .

Theorem 2.1 [13] *Let G be a graph and $k \geq 3$. Deciding whether G is k -colorable is an NP-complete problem.*

A graph G is **perfect** if and only if G is $\{C_{2n+1}, \bar{C}_{2n+1}, \text{ with } n \geq 2\}$ -free [2].

Theorem 2.2 [10] *Let G be a perfect graph and $k \geq 3$. Deciding whether G is k -colorable is a polynomial time solvable problem.*

3 Characterization of $[h, 2, 1]$, for $h \geq 3$

In this section we present a characterization of the VPT graphs that can be represented in a tree with maximum degree at most h . The characterization is given in terms of the chromatic number of the branch graphs. The following three lemmas are fundamental tools in the proof of the main Theorems 3.1 and 3.2.

Lemma 3.1 *Let $\langle \mathcal{P}, T \rangle$ be a VPT representation of G . Let $C \in \mathcal{C}(G)$ and $q \in V(T)$ such that $C = C_q$. If $v \in V(B(G/C))$ then P_v is contained in some branch of T at q . If v is adjacent to w in $B(G/C)$ then P_v and P_w are not contained in a same branch of T at q .*

Proof: If $v \in V(B(G/C))$ then $v \notin C$. It follows that $q \notin V(P_v)$, thus P_v is contained in some branch T_i of T at q . Let $w \in N_{B(G/C)}(v)$ and assume for a contradiction that P_v and P_w are contained in the same branch T_i . Let x and y be the vertices of P_v and P_w respectively at minimum distance from q . Since there exists a vertex of C adjacent to v and w , there exists a path in T containing q , x and y . We can assume, without loss of generality, that x is between q and y or that $x = y$. In both cases, $N(w) \cap C \subseteq N(v) \cap C$. This contradicts the fact that v and w are adjacent in the branch graph. \square

Lemma 3.2 *Let $\langle \mathcal{P}, T \rangle$ be a VPT representation of G . Let $C \in \mathcal{C}(G)$ and $q \in V(T)$ such that $C = C_q$. If $d_T(q) = h$, then $B(G/C)$ is h -colorable.*

Proof: Let T_1, T_2, \dots, T_h be the branches of T at q . By Lemma 3.1, if we color each vertex v of $B(G/C)$ with the index i of the branch T_i containing P_v , then we obtain a proper coloring of $B(G/C)$. Since there are h branches, $B(G/C)$ is h -colorable. \square

Lemma 3.3 *Let $\langle \mathcal{P}, T \rangle$ be a VPT representation of G . Consider $q \in V(T)$ with $d_T(q) = h \geq 4$. Assume there exist $y_1, y_2 \in N_T(q)$ such that for all $v \in V(G)$, $\{y_1, y_2\} \not\subseteq V(P_v)$. Then there exists a VPT representation $\langle \mathcal{P}', T' \rangle$ of G with $V(T') = V(T) \cup \{a_q\}$, $a_q \notin V(T)$, and*

$$d_{T'}(x) = \begin{cases} 3, & \text{if } x = a_q \\ h-1, & \text{if } x = q \\ d_T(x), & \text{if } x \in V(T') \setminus \{q, a_q\}. \end{cases}$$

Proof: We obtain the $\langle \mathcal{P}', T' \rangle$ representation of G as follows (Please refer to Figure 1): the set of vertices of T' is $V(T) \cup \{a_q\}$, where a_q is a new vertex not in $V(T)$. The set of edges is $(E(T) \setminus \{y_1q, y_2q\}) \cup \{y_1a_q, y_2a_q, qa_q\}$. Observe that the degree of each vertex $x \in V(T')$ is the required in the statement of the present lemma.

Now we define the paths P'_v for $v \in V(G)$: if y_1 and q or y_2 and q belong to $V(P_v)$ then $V(P'_v) = V(P_v) \cup \{a_q\}$. In any other case, $V(P'_v) = V(P_v)$. Since $\{y_1, q, y_2\} \not\subseteq V(P_v)$, we have that each $V(P'_v)$ induces a path in T' . Moreover, since all the paths where vertex a_q was added had vertex q in common, it is clear that, for any pair of vertices $v, w \in V(G)$, $V(P_v) \cap V(P_w) \neq \emptyset$ if and only if $V(P'_v) \cap V(P'_w) \neq \emptyset$. It follows that $\langle \mathcal{P}', T' \rangle$ is a VPT representation of G and the implication is proven. \square

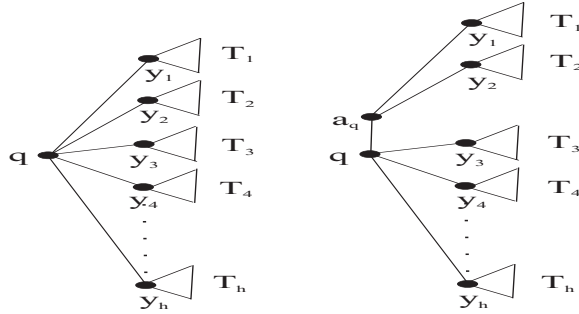


Figure 1: The degree of q in the tree T on the left is h . The degree of q in the tree T' on the right is $h-1$.

Theorem 3.1 *Let $G \in VPT$ and $h \geq 3$. The graph G belongs to $[h, 2, 1]$ if and only if $B(G/C)$ is h -colorable for every $C \in \mathcal{C}(G)$. The direct implication is true also for $h = 2$.*

Proof: Let $\langle \mathcal{P}, T \rangle$ be an $(h, 2, 1)$ -representation of G with $h \geq 2$. Assume $C \in \mathcal{C}(G)$, then there exists $q \in V(T)$ such that $C = C_q$. Since $d_T(q) \leq h$, by Lemma 3.2, $B(G/C)$ is h -colorable.

The reciprocal implication for $h = 3$ was proven by Golumbic and Jamison in [5]; then we assume $h \geq 4$.

Let $\langle \mathcal{P}, T \rangle$ be a VPT representation of G . We will prove that G admits an $(h, 2, 1)$ -representation.

We proceed by induction on the number k of vertices of T whose degree exceeds h . If $k = 0$ we are done.

If $k > 0$, there exists a vertex q of T with degree $d > h$. Say $N_T(q) = \{y_1, y_2, \dots, y_d\}$ and for every i , $1 \leq i \leq d$, let T_i be the branch of T at q containing the vertex y_i .

If by repeatedly applying the Lemma 3.3 we can obtain a VPT representation $\langle P', T' \rangle$ of G with $d_{T'}(q) < h$, then the implication is proven by induction since no vertex of T' increases its degree.

In other case, we can assume that for any pair of vertices $y_i, y_{i'}$ belonging to $N_T(q)$, there exists at least one $v \in V(G)$ such that $\{y_i, y_{i'}\} \subseteq V(P_v)$.

Notice that this implies that $C_q = \{v \in V(G) / q \in V(P_v)\}$ is a clique of G .

We will consider two cases.

Case 1: for every i , $1 \leq i \leq d$, there exists $v_i \in V(G)$ such that P_{v_i} is totally contained in the branch T_i and $y_i \in V(P_{v_i})$. Observe that each v_i must be a vertex of $B(G/C_q)$. Since $B(G/C_q)$ is h -colorable, we can partitioned the set $\{y_1, y_2, \dots, y_d\}$ in h subsets D_j , $1 \leq j \leq h$, each one containing the vertices y_i for which the associated vertex v_i has color j .

We obtain a new VPT representation $\langle P', T' \rangle$ of G as follows. The tree T' is obtained from T by means of the following procedure (in Figure 2 we offer an example): 1) remove the edges qy_i , $1 \leq i \leq d$; 2) add h new vertices μ_j , $1 \leq j \leq h$; 3) add the edges $q\mu_j$, $1 \leq j \leq h$; and finally, to connect the vertices μ_j with the vertices y_i , 4) add for every j , $1 \leq j \leq h$, a binary tree rooted at the vertex μ_j and with the vertices of D_j as leaves. The rest of the tree T remains unchanged.

The only paths of \mathcal{P} which are modified to obtain the paths of \mathcal{P}' are those $Q \in \mathcal{P}[q]$. If Q has q as an endpoint, then we obtain Q' by replacing in Q the edge qy_i by the unique subpath of T' linking q and y_i . If Q has q as an internal vertex, then there exist i and i' such that Q contains the edges qy_i

and $qy_{i'}$. Notice that the existence of Q implies that v_i and $v_{i'}$ are adjacent in $B(G/C_q)$; thus they have different colors, say j and j' . Therefore, we obtain Q' by replacing in Q the edges qy_i and $qy_{i'}$ by the only subpath of T' linking y_i , q and $y_{i'}$.

It is easy to see that this construction leaves the intersection graph of paths unchanged while reducing the number of tree vertices of degree greater than h . So, by induction, the implication is proven.

Case 2: there exists i , $1 \leq i \leq d$, such that every path $P \in \mathcal{P}$ containing y_i is not contained in the branch T_i . Thus, every path $P \in \mathcal{P}$ containing y_i contains also q . Therefore, we can contract the edge qy_i to obtain a new VPT representation of G and repeat this as many times as needed to get a representation which is in Case 1. Notice that in this procedure some vertices of T disappear, and that the degree of q may increase, but the number of vertices whose degree exceeds k does not grow, thus the proof follows by induction as in the previous case. \square

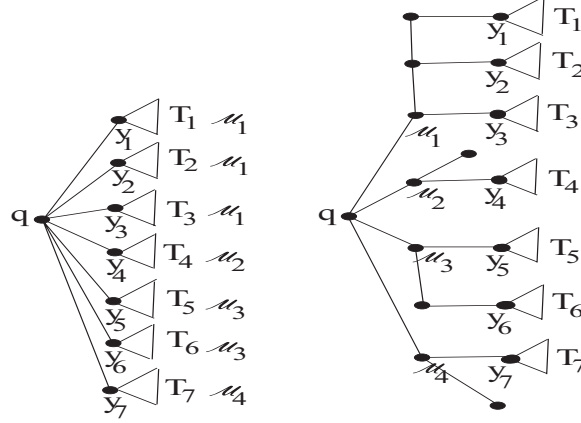


Figure 2: $d_T(q) = 7$ and $B(G/C_q)$ is 4-colorable.

Observe that the reciprocal implication of Theorem 3.1 is false for $h = 2$; consider, by instance, the graph T_2^3 which consists of one central vertex and 3 edge disjoint paths of 2 edges each intersecting only on the central vertex. It is easy to see that $T_2^3 \in VPT$ and $B(G/C)$ is 2-colorable for all $C \in \mathcal{C}(G)$, but $T_2^3 \notin [2, 2, 1]$.

Theorem 3.2 *Let $G \in VPT$ and $h \geq 4$. The graph G belongs to $[h, 2, 1] - [h - 1, 2, 1]$ if and only if $\text{Max}_{C \in \mathcal{C}(G)}(\chi(B(G/C))) = h$. The reciprocal implication is also true for $h = 3$.*

Proof: By Theorem 3.1, $G \in [h, 2, 1]$ if and only if $\text{Max}_{C \in \mathcal{C}(G)}(\chi(B(G/C))) \leq h$. On the other hand, by the same Theorem 3.1, $G \notin [h-1, 2, 1]$ if and only if $\text{Max}_{C \in \mathcal{C}(G)}(\chi(B(G/C))) > h-1$. Therefore, the proof follows. \square

4 Complexity

In this Section we prove that the problem of deciding whether a given graph belongs to $[h, 2, 1]$ for $h \geq 3$ is NP-complete. We also show that recognizing $[h, 2, 1] - [h-1, 2, 1]$ for $h \geq 4$ is NP-hard. Our results prove that both problems remain difficult even when restricted to the class $VPT \cap Split$ without dominated stable vertices.

First we state the following fundamental property of $VPT \cap Split$ graphs which is used in the proof of Theorems 4.1 and 4.2.

Lemma 4.1 *Let s be a stable vertex of a $VPT \cap Split$ graph G . The branch graph $B(G/N[s])$ is 1-colorable.*

Proof: Let $\langle P, T \rangle$ be a VPT representation of G such that P_s is a one vertex path in a leaf y of T , in other words $V(P_s) = \{y\}$ where y is a leaf of T . Thus $N[s]$ is the clique C_y . Since $d_T(y) = 1$, by Lemma 3.2, $B(G/N[s])$ is 1-colorable. \square

For the NP-completeness proof, we use a reduction from the chromatic number problem [13].

Given a graph G we construct in polynomial time a graph $\hat{G} \in VPT \cap Split$ without dominated stable vertices, in such a way that $\chi(G) = h$ if and only if $\hat{G} \in [h, 2, 1] - [h-1, 2, 1]$.

Let $V(G) = \{v_1, v_2, \dots, v_n\}$, we define the graph \hat{G} by means of its VPT representation $\langle \mathcal{P}, T \rangle$ as follows: the tree T is a star with central vertex q and leaves y_i for $1 \leq i \leq n$.

The path family \mathcal{P} contains: a one vertex path P_i with $V(P_i) = \{y_i\}$, for each $1 \leq i \leq n$; a path P_{ij} with $V(P_{ij}) = \{y_i, q, y_j\}$, for each $1 \leq i < j \leq n$ such that $v_i v_j \in E(G)$; a path P_{iq} with $V(P_{iq}) = \{q, y_i\}$, for each $1 \leq i \leq n$ such that $d_G(v_i) = 1$.

We call each vertex of \hat{G} as the corresponding path of \mathcal{P} .

In Figure 3 we offer an example of a graph G , the VPT representation of \hat{G} and the graph \hat{G} obtained.

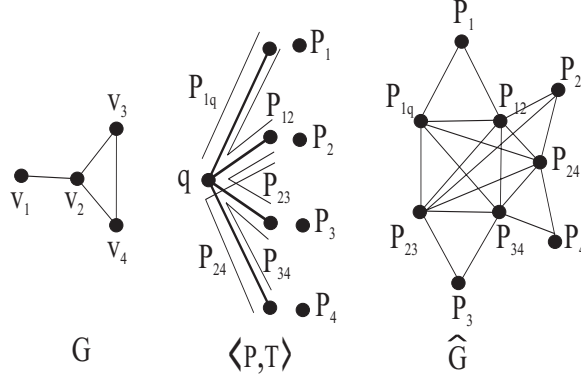


Figure 3: A graph G , the VPT representation of \widehat{G} and the graph \widehat{G} .

Notice that \widehat{G} is a split graph with the vertex set partitioned in a stable set of size $n = |V(G)|$ corresponding to the one vertex paths P_i ; and a central clique of size $|E(G)| + |\{v \in V(G) / d_G(v) = 1\}|$ corresponding to the remaining paths, all of which contain the vertex q of T , thus this clique is C_q . The other cliques of \widehat{G} are the cliques C_{y_i} for $1 \leq i \leq n$ each one corresponding to the paths containing the vertex y_i of T respectively. The graph \widehat{G} has no more cliques. In addition, every stable vertex P_i of \widehat{G} is non-dominated.

Observe that the branch graphs $B(\widehat{G}/C_{y_i})$ are described in Lemma 4.1, the following claim does for $B(\widehat{G}/C_q)$.

Claim 4.1 *If \widehat{G} is the graph obtained from G as above, then $B(\widehat{G}/C_q) = G$.*

Proof: Notice that $B(\widehat{G}/C_q)$ has exactly n vertices: P_i for $1 \leq i \leq n$.

We will see that P_i and P_j are adjacent in $B(\widehat{G}/C_q)$ if and only if v_i and v_j are adjacent in G . If $P_i P_j \in E(B(\widehat{G}/C_q))$ then there exists a vertex of C_q adjacent to both P_i and P_j . Then, there exists a path $P_{ij} \in \mathcal{P}$, thus $v_i v_j \in E(G)$. Reciprocally, let $v_i v_j \in E(G)$. Notice that P_i and P_j are non-adjacent in \widehat{G} ; and P_{ij} is a vertex of C_q adjacent to P_i and to P_j in \widehat{G} . Let us see that there exists a vertex of C_q adjacent to P_i and non-adjacent to P_j . Indeed, if $d_G(v_i) = 1$ then the wanted vertex of C_q is P_{iq} . If $d_G(v_i) > 1$ then v_i must have a neighbor v_l with $l \neq j$, thus the wanted vertex of C_q is P_{il} . In an analogous way, there exists a vertex of C_q adjacent to P_j and non-adjacent to P_i . We have proved that P_i and P_j are adjacent in $B(\widehat{G}/C_q)$. We conclude that $B(\widehat{G}/C_q) = G$. \square

The reduction from chromatic number is complete using the next claim.

Claim 4.2 *Let \widehat{G} be the graph obtained from G as above and $h \geq 4$. The graph \widehat{G} belongs to $[h, 2, 1] - [h - 1, 2, 1]$ if and only if $\chi(G) = h$.*

Proof: By Lemma 4.1 and Claim 4.1, $\max_{C \in \mathcal{C}(\widehat{G})} \chi(B(\widehat{G}/C)) = \chi(B(\widehat{G}/C_q)) = \chi(G)$. Hence, by Theorem 3.2, \widehat{G} belongs to $[h, 2, 1] - [h - 1, 2, 1]$ if and only if $\chi(G) = h$. \square

We have proved the following theorem.

Theorem 4.1 *Let $G \in VPT \cap Split$ without dominated stable vertices, and $h \geq 4$. Decide whether $G \in [h, 2, 1] - [h - 1, 2, 1]$ is an NP-hard problem.*

In addition, since an $(h, 2, 1)$ -representation is a polynomial certificate of belonging to $[h, 2, 1]$; using Theorem 3.1 and the construction above, we have proved the following result.

Theorem 4.2 *Let $G \in VPT \cap Split$ without dominated stable vertices, and $h \geq 3$. Decide whether $G \in [h, 2, 1]$ is an NP-complete problem.*

We notice that Theorem 4.2 for $h = 3$ has been previously proved in [5].

4.1 A polynomial time solvable subclass

We have proved that deciding whether a given $VPT \cap Split$ graph without dominated stable vertices admits a representation as intersection of paths of a tree with maximum degree h is an NP-complete problem. In what follows we describe a non-trivial subclass of $VPT \cap Split$ without dominated stable vertices where the problem is polynomial time solvable.

For $n \geq 4$, a **n-sun**, denoted by $\mathbf{S_n}$, is a split graph with stable set $\{s_1, s_2, \dots, s_n\}$, central clique $\{v_1, v_2, \dots, v_n\}$, $N(s_i) = \{v_i, v_{i+1}\}$ for $1 \leq i \leq n - 1$, and $N(s_n) = \{v_n, v_1\}$. See Figure 4.

Let G be a split graph with partition (S, K) . We say that G belongs to SVS (special VPT subclass) whenever

- $G \in VPT$,
- for all $v \in K$, $|N(v) \cap S| \leq 2$, and
- if S_k , with $k \in \{4, 2n + 1 \text{ for } n \geq 2\}$, is induced in G then there exists $v \in K$ such that v is adjacent to two non-consecutive vertices of the stable set of S_k .

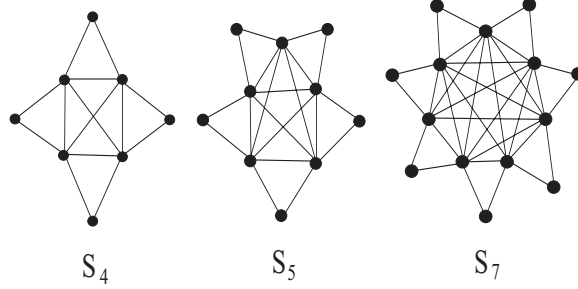


Figure 4: The sun graphs S_4 , S_5 and S_7 .

The class SVS is not trivial, in the sense that it includes graphs in $[h, 2, 1]$ for all $h \geq 4$.

For example, let $n \geq 4$ and let A_n (see [7]) be a split graph with partition (S, K) , where $S = \{s_1, \dots, s_n\}$, $K = \{v_{ij}, 1 \leq i < j \leq n\}$ and $N(v_{ij}) = \{s_i, s_j\}$, for all $1 \leq i < j \leq n$. It is clear that A_n belongs to SVS , and $B(A_n/K) = K_n$ with $V(K_n) = \{s_1, \dots, s_n\}$. Hence, by Theorem 3.2, $A_n \in [n, 2, 1] - [n-1, 2, 1]$. (As an example see Figure 5).

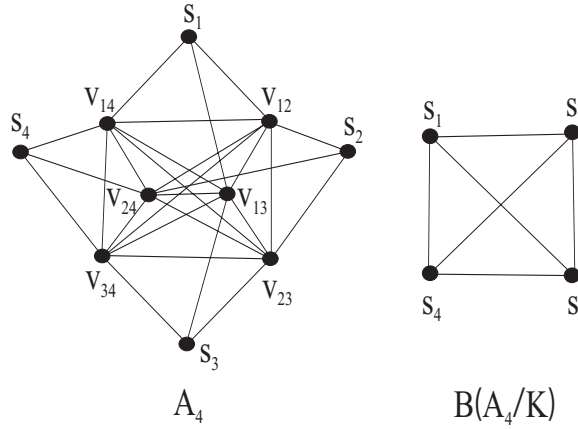


Figure 5: The graph A_4 belongs to SVS and $A_4 \in [4, 2, 1] - [3, 2, 1]$.

The following two lemmas are used in the proof of the main Theorem 4.3 which proves that in the class SVS the graphs belonging to $[h, 2, 1]$ can be recognized efficiently.

Lemma 4.2 *Let $G \in VPT \cap Split$ with partition (S, K) such that for all $v \in K$, $|N(v) \cap S| \leq 2$, and let $n \geq 4$. If $B(G/K)$ has an induced C_n then G has an induced S_n .*

Proof: Let $\langle \mathcal{P}, T \rangle$ be a *VPT* representation of G and $q \in V(T)$ such that $K = C_q$. Let C_n be an induced cycle of $B(G/K)$ with vertices s_1, s_2, \dots, s_n . It is clear that every $s_i \in S$. Since s_i is adjacent to s_{i+1} in $B(G/K)$, there exists $v_i \in K$ such that v_i is adjacent to s_i and to s_{i+1} in G . Since, for all $v \in K$, $|N(v) \cap S| \leq 2$, if $i \neq i'$ then $v_i \neq v_{i'}$, thus $s_1, s_2, \dots, s_n, v_1, v_2, \dots, v_n$ induce a n -sun in G and the proof is completed. \square

Lemma 4.3 *If $G \in SVS$ then every branch graph of G is perfect.*

Proof: Let (S, K) be a split partition of G . By Lemma 4.1, if $s \in S$ then $B(G/N[s])$ is perfect. Assume for a contradiction that $B(G/K)$ is not perfect, then $B(G/K)$ has induced an odd cycle or the complement of an odd cycle. Since the complement of C_5 is C_5 ; and the complement of any odd cycle of size 7 or more has an induced C_4 , it follows that $B(G/K)$ has an induced C_k , for some $k \in \{4, 2n+1 \text{ for } n \geq 2\}$. Therefore, by Lemma 4.2, G has an induced S_k . Since $G \in SVS$, there exists $v \in K$ such that v is adjacent to two non-consecutive vertices s and s' of the stable set of S_k . Notice that the existence of v implies that the vertices s and s' are adjacent in $B(G/K)$. This contradicts the fact that C_k is an induced cycle of $B(G/K)$. \square

Theorem 4.3 *Let $G \in SVS$ and $h \geq 4$. Decide whether G belongs to $[h, 2, 1] - [h-1, 2, 1]$ is polynomial time solvable.*

Proof: Given $G \in SVS$, in order to determinate if $G \in [h, 2, 1] - [h-1, 2, 1]$, by Theorem 3.1, it is enough to calculate the chromatic number of $B(G/K)$, where K is the central clique of G . Notice that the branch graph $B(G/K)$ can be obtained in polynomial time. On the other hand, by Lemma 4.3, $B(G/K)$ is perfect. Thus, by Theorem 2.2, its chromatic number can be calculated in polynomial time. \square

5 Future work

In this paper we give a characterization of the $[h, 2, 1]$ graphs, with $h \geq 3$. In addition, we prove that recognizing this class is NP-complete and show a family, called *SVS*, in which this problem is polynomial time solvable. We are working in describing a larger subclass of *VPT* graphs where this

problem remains polynomial. On the other hand, we are analyzing the possibility of extending the techniques used in the present paper to characterize the classes $[h, 2, 2]$.

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